2.3 Mixing Problems and Cooling Problems

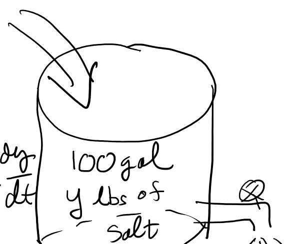
Example 2.3.1. A tank originally contains 100 gal of fresh water. Then water containing $\frac{1}{2}$ lb of salt per gallon is poured into the tank at a rate of 2 gal/min, and the mixture is allowed to leave at the same rate. After 10 minutes the process is stopped, and fresh water is poured into the tank at a rate of 2 gal/min, with the mixture again leaving at the same rate. Find the amount of salt in the tank at the end of an additional 10 min.

$y(t)$ = lbs of salt at time t.

2 important parts
1. picture
2. units

$y' = \frac{dy}{dt} = \frac{\text{lbs}}{\text{min}}$

$\frac{1}{2}\frac{\text{lb}}{\text{gal}}$, $\frac{2\text{ gal}}{\text{min}}$



$\frac{\text{lb}}{\text{gal}} \cdot \frac{\text{gal}}{\text{min}} = \frac{\text{lb}}{\text{min}} = \frac{dy}{dt}$

$\frac{2\text{ gal}}{\text{min}}$, $\frac{y}{100}\frac{\text{lb}}{\text{gal}}$

$y(t)$ = lbs of salt

$y(0) = 0$ lbs of salt (Fresh water)

Rate of Change = Rate in − Rate out

$\frac{\text{lb}}{\text{min}} \quad \frac{dy}{dt} = \left(\frac{1}{2}\frac{\text{lb}}{\cancel{\text{gal}}}\right)\left(\frac{2\cancel{\text{gal}}}{\text{min}}\right) - \left(\frac{2\cancel{\text{gal}}}{\text{min}}\right)\left(\frac{y}{100}\right)\frac{\text{lb}}{\cancel{\text{gal}}}$

$y' = 1 - \frac{y}{50}, \quad y(0) = 0$

Integrating factor

$u = e^{\int \frac{1}{50} dt} = e^{\frac{1}{50}t}$

$\int e^{\frac{1}{50}t}\left(y' + \frac{1}{50}y\right) = 50\int e^{1/50\, t}\frac{1}{50}dt \qquad u = \frac{1}{50}t, \quad du = \frac{1}{50}dt$

$\left(e^{\frac{1}{50}t} y = 50e^{\frac{1}{50}t} + C\right)e^{-\frac{1}{50}t}$

$y = 50 + Ce^{-\frac{1}{50}t}$

$y(0) = 0 = 50 + C$

$C = -50$

For the 1st 10 min

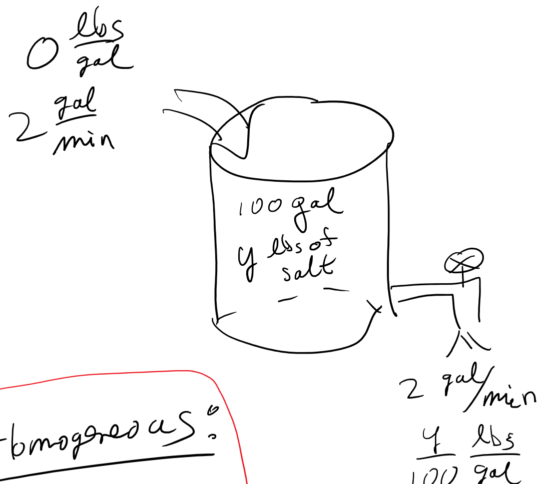
$$y_1 = 50 - 50e^{-\frac{1}{50}t}$$

How much salt in the tank after 10 min?

$y_1(10) = 9.06 \leftarrow$ Initial condition for part 2.

$y(t) =$ lbs of salt

$\rightarrow y(0) = 9.06$ lbs.



Rate = Rate in − Rate out

$$\frac{dy}{dt} = \left(0\,\frac{\text{lbs}}{\cancel{\text{gal}}}\right)\left(2\,\frac{\cancel{\text{gal}}}{\text{min}}\right) - \left(\frac{2\,\cancel{\text{gal}}}{\text{min}}\right)\left(\frac{y}{100}\,\frac{\text{lbs}}{\cancel{\text{gal}}}\right)$$

$y' = -\frac{y}{50}$, $\quad y(0) = 9.06$ lbs

Homogeneous:

$y' + p(t)y = 0$

$y = e^{-\int p(t)\,dt}$

$y' + \frac{1}{50}y = 0$

$-\int \frac{1}{50}\,dt = -\frac{1}{50}t + C$

$$y = Ce^{-\frac{1}{50}t}$$

$y(0) = 9.06 = C$

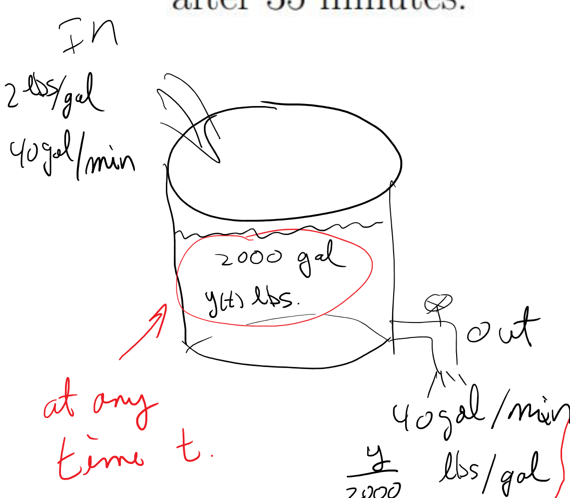
$$y_2 = 9.06e^{-\frac{1}{50}t}$$

$$\boxed{y_2(10) = 9.06e^{-\frac{1}{5}}} \text{ lbs of salt}$$

$= 7.42$ lbs

Example 2.3.2. In an oil refinery, a storage tank contains 2000 gallons of gasoline that initially has 100 lbs. of additive mixed in. In order to produce a different grade of gas, gasoline containing 2 lbs. of additive per gallon is pumped into the tank at the rate of 40 gallons per minute, and the well stirred mixture is pumped out at the same rate. Find the amount and concentration of additive after 35 minutes.

lbs. lbs/gal



$y(t)$ = amount of additive in tank

$y(0) = 100$ lbs.

Rate = Rate in − Rate out

$$\frac{\text{lbs}}{\text{min}} \quad \frac{dy}{dt} = 2(40) - \cancel{40}\left(\frac{y}{\cancel{2000}_{50}}\right)$$

$$\int e^{\frac{1}{50}t}\left(y' + \frac{1}{50}y\right) = (50)(80)\int e^{\frac{1}{50}t}\left(\frac{1}{50}dt\right)$$

$u = \frac{1}{50}t$
$du = \frac{1}{50}dt$

Integrating Factor:

$$u = e^{\int \frac{1}{50}dt} = e^{\frac{1}{50}t}$$

$$e^{-\frac{1}{50}t}\left(e^{\frac{1}{50}t}y = 4000e^{\frac{1}{50}t} + C\right)$$

$$y = 4000 + Ce^{-\frac{1}{50}t}$$

$$y(0) = 100 = 4000 + C$$

$$C = -3900$$

$$y = 4000 - 3900e^{-\frac{1}{50}t}$$

$$y(\infty) = \frac{2\text{ lbs}}{\text{gal}}(2000\text{gal}) = 4000\text{lbs}$$

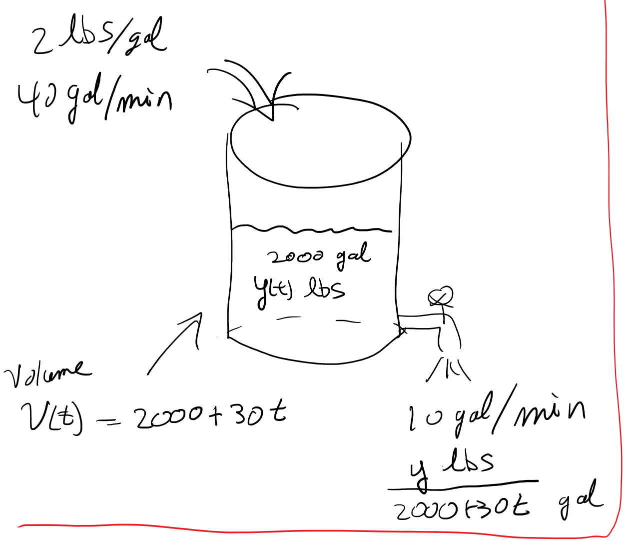
Same.

$$\lim_{t\to\infty} y(t) = 4000.$$

Amount is $y(35)$ lbs

Concentration is $\frac{y(35)}{2000}\frac{\text{lbs}}{\text{gal}}$

Example 2.3.3. In **Example** 2.3.2, suppose the tank holds 3000 gallons and is initially only 2/3 full of the original mixture. Gasoline containing 2 lbs. of additive per gallon flows into the tank at the rate of 40 gallons per minute but the well-stirred mixture is draining out at the slower rate of 10 gallons per minute. How many pounds of additive will be in the tank at the moment the tank becomes full?



$y(t)$ = lbs of additive in tank.

$y(0) = 100$

Rate = Rate in − Rate out

$$\frac{dy}{dt} = 80 - \frac{10y}{2000+30t}$$

$u = 200+3t$
$du = 3\,dt$

$$y' + \frac{y}{200+3t} = 80$$

$$\int (200+3t)^{1/3}\left(y' + \frac{y}{200+3t}\right) = \frac{80}{3}\int (200+3t)^{1/3}\,3\,dt$$

$$\frac{(200+3t)^{1/3}\,y}{(200+3t)^{1/3}} = \frac{\frac{80}{3}\frac{(200+3t)^{4/3}}{4/3} + C}{(200+3t)^{1/3}}$$

Int. Factor:

$$u = e^{\frac{1}{3}\int \frac{1}{200+3t}\,3\,dt}$$

$$= e^{\frac{1}{3}\ln(200+3t)}$$

$$= e^{\ln(200+3t)^{1/3}}$$

$$= (200+3t)^{1/3}$$

$$y = 20(200+3t) + C(200+3t)^{-1/3}$$

$$y(0) = 100 = 4000 + C(200)^{-1/3}$$

$$-3900(200)^{1/3} = C$$

$$C = -22807$$

$$y(t) = 20(200+3t) - 22807(200+3t)^{-1/3}$$

How much when tank is full?

Fill 1000 gal at 30 gal/min

$t = \frac{100\cancel{0}}{3\cancel{0}} = 33.3$ min

$y(33.3) = 2593$ lbs

$$\int e^{\frac{1}{50}t}\,dt = \int e^{u}\,50\,du$$

$u = \frac{1}{50}t$

$du = \frac{1}{50}dt$

$50\,du = dt$

$$= 50\int e^{u}\,du$$

$$= 50e^{u} + C$$

$$= 50e^{\frac{1}{50}t} + C$$