Math 279 Name:______________________________

Test #2 Spring 03

Chalmeta

DIRECTIONS: In order to receive credit for any answer, you must show all necessary work. You can answer the questions either on this sheet or on separate paper. If you use your own paper work one problem per page, number each problem as you work it and staple this sheet in front of your work. No cheating.

For the multiple choice problems you don't need to show any work. (4 pts each)

1. The 4th order differential equation $y^{(4)} - y''' + 8y'' - 8y' + 4y = 0$ has characteristic equation $(r-(1+i))^2(r-(1-i))^2 = 0$ in factored form. Which of the following is the general solution?
   a) $e^t(c_1 \cos t + c_2 \sin t) + e^t(c_3 \cos t + c_4 \sin t)$
   b) $e^t(c_1 \cos t + c_2 \sin t) + te^t(c_3 \cos t + c_4 \sin t)$
   c) $c_1 \cos t + c_2 \sin t + t(c_3 \cos t + c_4 \sin t)$
   d) $e^t(c_1 \cos t + c_2 \sin t) + e^t(c_3 \cos(-t) + c_4 \sin(-t))$

2. Let a < x < b denote the largest interval on which the solution of the following initial value problem exists.
   $$2(x-1)(x-2)x^2 y'' - 2xy' + (x-7)y = 0, \quad y(-1) = 4, \; y'(-1) = 0$$
   Then the interval a < x < b is

   a) $-2 < x < 0$
   b) $-5 < x < 1$
   c) $-5 < x < 2$
   d) $-\infty < x < 0$

3. Which one of the following pairs of functions is linearly independent?
   a) t, t + 1
   b) t/2, 2t
   c) e^t, $3e^t$
   d) e^t, $3e^{t-1}$

4. (10 pts) Find an appropriate form for y_p if the method of undetermined coefficients were used to solve each of the following.

   a) $y'' + 2y' + 2y = t^2 e^{-t} + 3e^{-t} \cos t$

   b) $y'' + 2y' + y = te^{-t} + 3e^{-t} \cos t$

5. (10 pts) Suppose a 32 lb weight is attached to a spring and stretches the spring 4 feet. The object is pulled down 6” and released. Find the coefficient of resistance for which the resulting motion would be critically damped, then determine the displacement from equilibrium at t = 3 sec.

6. (10 pts) One solution of $t^2 y'' - 3ty' + 3y = 0$ is $y = t$. Use reduction of order to find the general solution.

7. (10 pts) Find the general solution to the differential equation $y'' + 4y = \sec 2t$

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Solutions:

1. $y^{(4)} - y''' + 8y'' - 8y' + 4y = 0$ has characteristic equation

   $(r-(1+i))^2(r-(1-i))^2 = 0$

The solutions to the characteristic equation are

   $r = (1 \pm i),\ (1 \pm i)$

Since we have repeated roots we have a duplicate solutions

   $\boxed{y_h = e^t(c_1 \cos t + c_2 \sin t) + te^t(c_3 \cos t + c_4 \sin t)}$

2. The standard form of the differential equation is

   $y'' - \dfrac{2x}{2(x-1)(x-2)x^2}y' + \dfrac{(x-7)}{2(x-1)(x-2)x^2}y = 0, \quad y(-1) = 4,\ y'(-1) = 0.$

   Then the possible intervals are $-\infty < x < 0$, $0 < x < 1$, $1 < x < 2$, $2 < x < \infty$. Since our initial condition is at t = -1 chose $\boxed{-\infty < x < 0}$

3. $W(t, t+1) \neq 0$ so they are linearly independent.

4. Solve for y_h

   a) $y'' + 2y' + 2y = t^2e^{-t} + 3e^{-t}\cos t$

   $y_h = e^{-t}(C_1 \cos t + C_2 \sin t)$

   Our initial guess for the particular solution will duplicate this solution so we need to multiply by t.

   $\boxed{y_p = (A_1t^2 + A_2t + A_3)e^{-t} + te^{-t}(B_1 \cos t + B_2 \sin t)}$

   b) $y'' + 2y' + y = te^{-t} + 3e^{-t}\cos t$

   $y_h = e^{-t} + te^{-t}$

   Once again the initial guess duplicates the homogenous solution so multiply by t^2 because there are two terms to fix.

   $\boxed{y_p = t^2(A_1t + A_2)e^{-t} + e^{-t}(B_1 \cos t + B_2 \sin t)}$

5. m = 32/32 = 1, γ = ?, k = 32/4 = 8. Critical damping occurs when $\sqrt{g^2 - 4mk} = 0$. So $\boxed{g = \sqrt{4mk} = 4\sqrt{2}}$ and the equation becomes:

$u''+4\sqrt{2}u'+8u = 0, \quad u(0) = 1/2, \quad u'(0) = 0$

The solutions are $r = \dfrac{-4\sqrt{2}}{2} = -2\sqrt{2}$ (twice)

$u = C_1 e^{-2\sqrt{2}t} + C_2 t e^{-2\sqrt{2}t}$ and using the initial conditions $C_1 = 1/2, C_2 = \sqrt{2}$

$u(t) = (1/2)e^{-2\sqrt{2}t} + \sqrt{2}te^{-2\sqrt{2}t}$ and $\boxed{u(3) = .00098 \text{ ft.}}$

6. The second solution will be $y_2 = ut$ where u is to be determined. Take two derivatives: $y_2' = u + u't$ and $y_2'' = 2u' + u''t$. Substitute into the original equation: $t^2(2u'+tu'') - 3t(u+u't) + 3ut = 0 \Rightarrow t^3u'' - t^2u' = 0$ which is a first order equation in u' so $\displaystyle\int\frac{u''}{u'} = \int\frac{1}{t} \Rightarrow u' = t \Rightarrow u = \frac{t^2}{2}$. Our second solution is then $y_2 = \dfrac{t^3}{2}$ or simply $\boxed{y_2 = t^3}$.

7. $y''+4y = \sec 2t$ has characteristic equation $r^2 + 4 = 0$ and solutions $r = \pm 2i$. The homogeneous solution is $y_h = C_1\cos(2t) + C_2\sin(2t)$. To use variation of parameters replace C_1 and C_2 with u and v respectively.

$y_p = u\cos(2t) + v\sin(2t)$

$y_p' = u'\cos(2t) - 2u\sin(2t) + v'\sin(2t) + 2v\cos(2t)$

Impose the condition $0 = u'\cos(2t) + v'\sin(2t)$ **(1)**

$y_p' = -2u\sin(2t) + 2v\cos(2t)$

$y_p'' = -2u'\sin(2t) - 4u\cos(2t) + 2v'\cos(2t) - 4v\sin(2t)$

Putting it all back into the original equation gives:

$-2u'\sin(2t) + 2v'\cos(2t) = \sec(2t)$ **(2)**

Solve equation **(1)** for $u' = \dfrac{-v'\sin(2t)}{\cos(2t)}$ and substitute in **(2)** to get :

$$2v'\frac{\sin^2(2t)}{\cos(2t)} + 2v'\cos(2t) = \frac{1}{\cos(2t)}$$

$$2v'\sin^2(2t) + 2v'\cos^2(2t) = 1 \Rightarrow v' = 1/2 \quad and \quad v = t/2$$

$$u' = \frac{-\sin(2t)}{2\cos(2t)} \Rightarrow u = \frac{1}{4}\ln|\cos(2t)|$$

$$\boxed{y_p = \frac{1}{4}\ln|\cos(2t)|\cos(2t) + \frac{1}{2}t\sin(2t)}$$